Tutorial 1 : Selected problems of Assignment 1

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Q1) (Ex. 1 Q1)

Let $\mathcal{F} = \{\text{finite Fourier series}\}$

$$= \left\{ f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \right\}$$

$\mathcal{P} = \{\text{trigonometric polynomials}\}$

$$= \left\{ p(\cos x, \sin x) = \sum_{\substack{j,k \geq 0 \\ j+k \leq N}} a_{jk} \cos^j x \sin^k x \right\}$$

Show that $\mathcal{F} = \mathcal{P}$

Sol: Recall Euler formula: $e^{ix} = \cos x + i \sin x$

$\therefore \forall n \geq 0, \quad e^{inx} = \cos nx + i \sin nx$

On the other hand, by binomial theorem,

$$e^{inx} = (\cos x + i \sin x)^n = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k} x \, (i \sin x)^k$$

$= q_n(\cos x, \sin x) + i\, r_n(\cos x, \sin x)$, where $q_n, r_n \in \mathcal{P}$

$$\therefore \begin{cases} \cos nx = q_n(\cos x, \sin x) \in \mathcal{P} \\ \sin nx = r_n(\cos x, \sin x) \in \mathcal{P} \end{cases}$$

$\therefore \forall f \in \mathcal{F}, \quad f(x) = a_0 + \sum_{n=1}^{N} (a_n q_n + b_n r_n) \in \mathcal{P}$, Hence $\mathcal{F} \subseteq \mathcal{P}$.

Conversely: $\forall n \in \mathbb{N}$, $\cos^n x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n$

$$= \frac{1}{2^n}\left(\sum_{k=0}^{n} \binom{n}{k} e^{i(n-k)x} e^{-ikx}\right)$$

(where $c_k \in \mathbb{R}$)

$$= \sum_{k=0}^{n} c_k e^{i(n-2k)x} = \underbrace{\sum_{k=0}^{n} c_k \cos(n-2k)x}_{f_n(x)} \in \mathcal{F}$$

Similarly, $\sin^n(x) = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^n$

$$= \frac{1}{(2i)^k}\left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{i(n-k)x} e^{-ikx}\right)$$

(where $d_k, e_k \in \mathbb{R}$)

$$\boxed{= \underbrace{\sum_{k=0}^{n} \left(d_k \cos(n-2k)x + e_k \sin(n-2k)x\right)}_{g_n(x)} \in \mathcal{F}}$$

Corrected version

$\therefore \forall p \in \mathcal{P}$, $p(x) = \sum_{j,k} a_{jk} \cos^j x \sin^k x$

$$= \sum_{j,k} a_{jk} f_j(x) g_k(x) \in \mathcal{F}$$

(by product-to-sum formula)

Hence $\mathcal{P} \subseteq \mathcal{F}$

Combining above, we have $\mathcal{F} = \mathcal{P}$.

Q2) (Ex. 1 Q2)

Let $f : [-\pi, \pi] \to \mathbb{R}$ be 2π-periodic integrable, show that

(i) f is integrable over any finite closed interval

(ii) $\forall\, I, J \subseteq \mathbb{R}$ closed interval of length 2π,

$$\int_I f(x)\,dx = \int_J f(x)\,dx$$

Pf: (i) f is integrable over $[-\pi, \pi]$

$$\Downarrow \text{ (} f : 2\pi\text{-periodic)}$$

f is integrable over $[(2n-1)\pi, (2n+1)\pi]$, $\forall n \in \mathbb{Z}$

$$\Downarrow$$

f is integrable over $[-(2m+1)\pi, (2m+1)\pi]$, $\forall m \in \mathbb{Z}$

Then $\forall\, K \subseteq \mathbb{R}$ finite closed interval, there exists $N \in \mathbb{N}$ s.t.

$K \subseteq [-(2N+1), (2N+1)]$, $\therefore$ f is integrable over K.

(ii) It suffices to show that $\int_I f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx$:

Write $I = [a, a+2\pi]$, $\exists\, a \in \mathbb{R}$. Then $\exists\, n \in \mathbb{Z}$ s.t. $n\pi \in I$.

$$\therefore \int_I f(x)\,dx = \int_a^{n\pi} f(x)\,dx + \int_{n\pi}^{a+2\pi} f(x)\,dx$$

$$= \int_{a+2\pi}^{n\pi+2\pi} f(y)\,dy + \int_{n\pi}^{a+2\pi} f(x)\,dx$$ (by change of variable $y = x+2\pi$ on the first integral)

$$= \int_{n\pi}^{(n+2)\pi} f(y)\,dy = \int_{-\pi}^{\pi} f(x)\,dx$$ (by change of variable $x = y-(n+1)\pi$)

$$\therefore \int_I f(x)\,dy = \int_{-\pi}^{\pi} f(x)\,dx = \int_J f(x)\,dx$$

Q3) (Ex. 1, Q7)

Let $f : [-\pi, \pi] \to \mathbb{R}$ be 2π-periodic differentiable function

such that $f' : [-\pi, \pi] \to \mathbb{R}$ is integrable

Let $\begin{cases} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{cases}$ be the Fourier coefficients of f.

Show that $|a_n|, |b_n| \to 0$ as $n \to \infty$ without using Riemann-Lebesgue lemma.

Pf: Note that $\int_{-\pi}^{\pi} f'(x) \cos nx \, dx$

$$= \left[f(x) \cos nx \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x) (-n \sin nx) \, dx$$

$$= 0 + n b_n = n b_n$$

$(\because f' \text{ is bounded, } \|f'\|_\infty \leq M, \exists M \in \mathbb{R})$

$$\therefore |b_n| = \frac{1}{n} \left| \int_{-\pi}^{\pi} f'(x) \cos nx \, dx \right| \leq \frac{1}{n} \int_{-\pi}^{\pi} M \, dx \to 0$$

as $n \to \infty$

Similarly for a_n: $\int_{-\pi}^{\pi} f'(x) \sin nx \, dx = -n a_n$

$\therefore \quad |a_n| = \frac{1}{n} \left| \int_{-\pi}^{\pi} f'(x) \sin nx \, dx \right| \leq \frac{1}{n} \int_{-\pi}^{\pi} M \, dx \to 0$

as $n \to \infty$